Int. J. Solids Structures Vol. 28, No. 5, pp. 663–664, 1991
Printed in Great Britain.

0020-7683/91 $3.00 + .00

AUTHOR'S CLOSURE

1. ANALYSIS OF VALANIS' CRITICISMS

The criticisms raised by Prof. Valanis in his Letter to the Editor may be easily refuted by the two following remarks:

(1) Cases (a) and (c) are, of course, mutually exclusive, i.e. if Case (a) holds, (c) cannot hold, and vice versa. Valanis has inadvertently overlooked this fact when deriving (iii), which he asserts, hold for Case (a). Indeed, in the process of the derivation of (iii) he invokes (22), which corresponds to Case (c). Hence, (iii) is obviously an invalid result.

(2) The invalidity of (iii) notwithstanding, let us assume for a moment that (iii) were to hold, in which case (iv) would indeed be a solution to the differential equation treated in Section 3 of Fazio (1989); (iv), however, cannot possibly satisfy the initial conditions chosen for the latter, i.e. $\varepsilon = 0$, $\sigma = 0$. It should be emphasized that Section 3 of Fazio (1989) simply poses an initial-value problem, which represents a concrete physical situation, namely, that of the stress response to a loading from an unstrained state. Valanis' stress–plastic strain diagram shown in Fig. 1 is not only incorrect, as we have seen, but does not include the origin (0, 0), whence it cannot realistically predict the stress response to a loading from an unstrained configuration. In view of this fact, it is unwarranted to claim that the solution (iv) describes elastic material behavior, as Valanis implicitly does by calling σ_y the yield stress. To quote Prof. Valanis, this situation, the reader will agree, is somewhat unfortunate.

The possibility that the denominator in (24) might be equal to zero was duly considered in Fazio (1989). Indeed, it was stressed in that paper that the solutions to (24) are not continuous in the parameter κ, and explained that this was a consequence of the fact that $F(\sigma, \varepsilon, \kappa)$ has a (removable) discontinuity. The latter is, of course, situated at those points of the σ–ε plane where the denominator in (24) becomes zero, i.e. on the curve given by:

$$\sigma = \frac{E_0 f(\zeta)}{\lambda P(\varepsilon_p)} + E_\kappa \varepsilon$$

which encompasses (iii) as a special case. Since the above curve cannot possibly include the origin of the σ–ε plane, the Lipschitz condition is satisfied in a neighborhood of that point, which in turn assures the existence and uniqueness of the solution of the initial-value problem dealt with here. In view of the foregoing remarks, the initial-value problem posed in Fazio (1989) is rather trivial: the only possible solution of (24), which is reduced to $d\sigma = E_0\, d\varepsilon$ for $\kappa = 1$, satisfying the given initial conditions, is the elastic stress-response $\sigma = E_0 \varepsilon$.

I am grateful to Prof. Valanis, nonetheless, for pointing out that the above elastic stress response violates condition (a), on which the analysis leading to it was based. This is quite correct. I chose to postpone the rigor in proving the incapability of the endochronic theory to describe inelastic material behavior for the limit case $\kappa = 1$ until Section 5. It is evident that the case treated in Section 3 is simply a special case of the more general one dealt with in Section 5, where the above result is rigorously demonstrated by means of a theorem, whose proof follows by *reductio ad absurdum*. Therefore, one should consider the special case of Section 3 in the same spirit. And indeed, the assumption that $d\sigma > E_0\, d\varepsilon$ (always considering the aforementioned limit case) yields the same type of contradiction that permitted to prove the aforementioned theorem, and likewise shows that the only possible case is (c), i.e. $\sigma = E_0 \varepsilon$. To be more specific, if one assumes the former relationship, one obtains for the stress response a differential equation whose particular solution for the above initial conditions is given by the latter relationship, thus showing that the only admissible condition is (c). But in this case the sought for particular solution is again $\sigma = E_0\varepsilon$.

2. CONCLUSIONS

Although Prof. Valanis has addressed only one of the three analyses presented in Fazio (1989), he concludes that I have written an entire paper based on the division of zero by zero. Having literally ignored the two remaining analyses (presented in Section 4 and 5 of my paper, respectively), he promptly states, nevertheless, that all my results are "blatantly wrong". Despite this emphatic statement, the two remaining analyses only come to corroborate the result of Section 3 of Fazio (1989). Actually, Prof. Valanis has yet to show the aforementioned theorem wrong if he is intent in disproving the results advanced in that reference.

Unfortunately, everything does not fit and is far from being as simple as claimed by Prof. Valanis. On the contrary, the theory is replete with mathematical inconsistencies for the limit case $\kappa = 1$, as was demonstrated in Fazio (1990). But while one can go on at length in proving that the limit case $\kappa = 1$ of the endochronic theory leads to mathematical inconsistencies of all kinds, the crux of the matter lies, however, in the following simple reasoning of pure physical nature presented in Fazio (1990). If this theory were capable for the limit case $\kappa = 1$ of describing an elastic stress response at the onset of loading, as claimed by its author, then the constitutive equations of the theory would be inconsistent. Indeed, in that limit case the stress response becomes a functional of the plastic strain *alone*. Hence, in order for an elastic stress response to take place, plastic deformations must occur simultaneously, which is evidently a contradiction *per se*, however persistently Prof. Valanis may attempt to prove the contrary. It is this physical contradiction which should be regarded as the ultimate cause of all the mathematical contradictions of the theory ensuing for the limit case $\kappa = 1$. In brief, the attempt to obtain the classical theory of plasticity from the endochronic theory is simply ill-founded.

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